Q. Suppose $X_1, X_2, \cdots X_n$ are i.i.d. from the Exponential (λ) distribution ($E[X_i] = \frac{1}{\lambda}$), what is the method of moments estimator for λ?

Using the first moment

$$M_1 = \frac{\sum X_i}{n} = \bar{X} \qquad \text{Since } E[X_i] = \frac{1}{\lambda}$$

$$\Rightarrow \hat{\lambda}_{mom} = \frac{1}{\bar{X}}$$

Using the second moment

$$M_2 = \frac{\sum X_i^2}{n}$$

$$\text{Since } E[X_i^2] = V(X_i) + E(X_i)^2 = \frac{1}{\lambda^2} + \frac{1}{\lambda^2}$$

$$\Rightarrow \hat{\lambda}'_{mom} = \sqrt{\frac{2n}{\sum X_i^2}}$$

if we do many simulations (random generate enough x_i, and calculate $\hat{\lambda}_{mom}$ $\hat{\lambda}'_{mom}$, we can find that $E[\hat{\lambda}'_{mom}] > E[\hat{\lambda}_{mom}]$, but both are biased and $V[\hat{\lambda}'_{mom}] > V[\hat{\lambda}_{mom}]$

Q Adjust the MOM estimator to make it an unbiased estimator

If we know $\bar{X} \sim \text{Gamma}(n, n\lambda)$

we can claim $\frac{1}{\bar{X}} \sim \text{Inverse Gamma}(n, n\lambda)$

for $Y \sim \text{Inverse Gamma}(n, n\lambda)$, $E[Y] = \frac{\beta}{\alpha - 1}$, $\alpha > 1$

thus $E\left[\frac{1}{\bar{X}}\right] = \frac{n \cdot \lambda}{n-1} \neq \lambda$

define $\hat{\lambda}_3 = \frac{n-1}{n} \frac{1}{\bar{X}}$

$E[\hat{\lambda}_3] = \lambda$ and $V[\hat{\lambda}_3] = V\left[\frac{n-1}{n} \hat{\lambda}_{mom}\right] < \hat{\lambda}_{mom}$

therefore $\hat{\lambda}_3$ is a better estimator than the first moment's estimator.

Q: An urn contains black and white balls. All we know is that the balls have a color ratio of 1 to 3 (but we don't know which one is more) We draw three balls with replacement from the urn. Let X be the number of black draws. Derive the likelihood function and the MLE as a function of X.

$\theta = P$ (single draw is black) $\theta = \frac{1}{4}$ or $\frac{3}{4}$

$X \sim$ binomial $(3, \theta)$

	$X=0$	$X=1$	$X=2$	$X=3$
$\theta = \frac{1}{4}$	$27/64$	$27/64$	$9/64$	$1/64$
$\theta = \frac{3}{4}$	$1/64$	$9/64$	$27/64$	$27/64$

$$\hat{\theta}_{MLE} = \begin{cases} \frac{1}{4} & X = 0, 1 \\ \frac{3}{4} & X = 2, 3 \end{cases}$$

Q. Now suppose θ equals the proportion of balls in the urn which are black. We know that $0<\theta<1$. We draw ten times with replacement, and count seven black balls and three white balls. Derive the MLE for θ

Let X be the number of black draws.

$X \sim \text{binomial}(10, \theta)$

$$f_X(x) = \binom{10}{x} \theta^x (1-\theta)^{10-x} \quad , x = 0, 1, \cdots 10$$

The likelihood function when $x=7$ is

$$L(\theta) = \binom{10}{7} \theta^7 (1-\theta)^3$$

$$\Rightarrow \frac{\partial L(\theta)}{\partial \theta} = \binom{10}{7} \cdot \left[7\theta^6 (1-\theta)^3 - 3\theta^7 (1-\theta)^2 \right] = 0$$

$$\Rightarrow \theta = 0.7$$

Q: Suppose $X_1, X_2, \cdots X_n$ are i.i.d. with Exponential(λ) distribution. Derive the MLE for λ

$$f_{x_i}(x_i) = \lambda e^{-\lambda x_i} \quad x_i > 0$$

$$L(\lambda) = \prod_{i=1}^{n} f_{x_i}(x_i) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i} \quad x_i > 0$$

$$= \lambda^n e^{-\lambda \Sigma x_i} \quad x_i > 0$$

$$l(\lambda) = \log L(\lambda) = n \log \lambda - \lambda \Sigma x_i$$

$$\frac{\partial l(\lambda)}{\partial \lambda} = \frac{n}{\lambda} - \Sigma x_i = 0 \Rightarrow \lambda = \frac{n}{\Sigma x_i} = 1/\bar{x}$$

Q. Suppose $X_1, X_2, \cdots X_n$ are i.i.d. with the Beta(α,1) distribution. Derive the MLE for α.

$$f_{X_i}(x_i) = \alpha x_i^{\alpha-1} \qquad 0 < x_i < 1$$

$$L(\alpha) = \prod_{i=1}^{n} f_{X_i}(x_i) = \alpha^n \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} \qquad 0 < x_i < 1$$

$$l(\alpha) = \log L(\alpha) = n \log \alpha + (\alpha - 1) \cdot \sum_{i=1}^{n} \log x_i$$

$$\frac{\partial l(\alpha)}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log x_i = 0 \Rightarrow \alpha = -\frac{n}{\sum_{i=1}^{n} \log x_i}$$

Recall the MoM estimator :

$$E[X_i] = \frac{\alpha}{\alpha+1} \Rightarrow \mu_1 = \bar{x} \qquad \bar{x} = \frac{\alpha}{\alpha+1} \Rightarrow \hat{\alpha}_{mom} = \frac{\bar{x}}{1-\bar{x}}$$

Q. Assume $X_1, X_2, \cdots X_n$ are i.i.d. with the Normal(μ, σ^2) distribution. Derive the MLE for μ and σ^2.

$$f_{x_i}(x_i) = \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{(x_i-\mu)^2}{2\sigma^2}} \qquad -\infty < x_i < \infty$$

$$L((\mu, \sigma^2)) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n e^{-\frac{\sum (x_i-\mu)^2}{2\sigma^2}}$$

$$l(\mu, \sigma^2) = -\frac{n}{2} \log 2\pi - \frac{n}{2} \log \sigma^2 - \frac{1}{2\sigma^2} \sum (x_i - \mu)^2$$

$$\frac{\partial l}{\partial \mu} = \frac{1}{\sigma^2} \sum (x_i - \mu) = 0 \Rightarrow \mu_{MLE} = \bar{x}$$

$$\frac{\partial L}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum (x_i - \mu)^2 = 0 \Rightarrow \sigma^2 = \frac{\sum (x_i-\mu)^2}{n} \neq s^2$$

Q. Suppose $X_1, X_2, \cdots X_n$ are i.i.d. Uniform$(0, \theta)$. Derive the MLE for θ. Derive the MOM for θ.

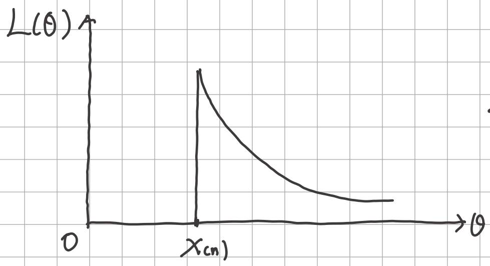
$$f_{X_i}(x_i) = \frac{1}{\theta} \cdot \mathbb{1}_{\{0 < x_i < \theta\}}$$

$$L(\theta) = \prod_{i=1}^{n} f_{X_i}(x_i) = \left(\frac{1}{\theta}\right)^n \prod_{i=1}^{n} \mathbb{1}_{\{0 < x_i < \theta\}}$$

$$= \left(\frac{1}{\theta}\right)^n \mathbb{1}_{\{X_{(n)} < \theta\}} \cdot \mathbb{1}_{\{X_{(1)} > 0\}}$$

where $X_{(n)} = \max\{X_1, X_2, \cdots X_n\}$

$X_{(1)} = \min\{X_1, X_2, \cdots X_n\}$



thus $\underbrace{\hat{\theta}_{MLE} = X_{(n)}}_{\text{can be extremely biased}}$

MOM estimator : $E[X_i] = \theta/2 \quad M_1 = \bar{X} \Rightarrow \theta = \overset{\text{unbiased}}{2\bar{X}}$

however, note that there is no guarantee that

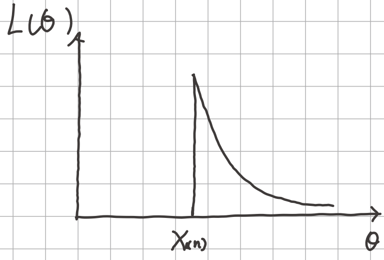
$$2\bar{X} > X_{(n)}$$

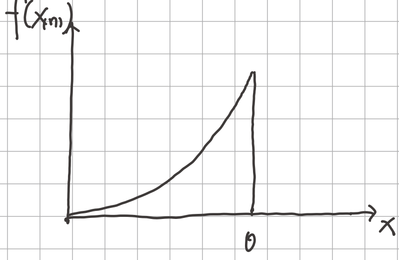
Furthurmore, we can compare $L(\theta)$ with the distribution of $X_{(n)}$.

$$P(X_{(n)} \leq x) = P(X_1 \leq x, X_2 \leq x, \cdots X_n \leq x)$$

$$= \left(\frac{x}{\theta}\right)^n$$

$$f(X_{(n)}) = n\left(\frac{x}{\theta}\right)^{n-1} \quad 0 < x < \theta$$





Note that as $n \nearrow +\infty$, the shape of $f(X_{(n)})$ remains the same, and it's never approximating normal. That is because we are actually violating one of the regularity conditions, namely that support is not the same for all θ.