Q Suppose that $X_1, X_2, \cdots X_n$ are i.i.d. Exponential(λ).
We seek to estimate $\tau = P(X_i > a)$ for some constant $a > 0$
Construct the MLE and find its standard error. Also construct a confidence interval for τ.

By invariance property. $\hat{\tau}_{MLE} = e^{-a/\bar{x}}$ i.e. $g(\tau) = e^{-a\lambda}$

We know that $I(\lambda) = 1/\lambda^2$, use the delta method,

$g'(\lambda) = -ae^{-a\lambda}$, thus $\hat{\tau}_{MLE} = g(\hat{\tau}_{MLE})$ is approximately normal with mean $g(\lambda_0) = e^{-a\lambda_0}$ and variance

$$(g'(\lambda_0))^2 \left(\frac{1}{nI(\lambda_0)}\right) = a^2 e^{-2a\lambda} \left(\frac{\lambda_0^2}{n}\right)$$

In practical terms, $\hat{\tau}_{MLE}$ is approximately normal with Variance $a^2 e^{-2a\lambda}\left(\frac{1}{n\bar{x}^2}\right)$, and the approx SE is

$$\hat{SE} = (ae^{-a\lambda}) \frac{1}{\sqrt{n}\,\bar{x}}$$

A $100(1-\alpha)\%$ CI for τ is

$$\hat{\tau} \pm Z_{\frac{\alpha}{2}} \hat{SE}.$$

or $$P\left(-Z_{\frac{\alpha}{2}} \le \frac{\hat{\tau}_{MLE} - \tau_0}{\hat{SE}} \le Z_{\frac{\alpha}{2}}\right) = 1-\alpha$$