Q_1 let $0 \le s < t$ be given, what is $Cov[W(s), W(t)]$

A_1 $W(t) = \big(W(t) - W(s)\big) + \big(W(s) - W(0)\big)$
$\quad = \big(W(t) - W(s)\big) + W(s)$
standard trick

$W(t) - W(s)$ and $W(s)$ are independent

$W(t) - W(s) \sim N(0, t-s)$ $\quad W(s) \sim N(0, s)$

$$
\begin{aligned}
Cov[W(s), W(t)] &= Cov[W(s), W(t) - W(s) + W(s)] \\
&= Cov[W(s), W(t) - W(s)] + Var(W(s)) \\
&= E[W(s) \cdot (W(t) - W(s))] + s \\
&= E[W(s)] \cdot E[W(t) - W(s)] + s \\
&= 0 \times 0 + s \\
&= s
\end{aligned}
$$

Without the assumption $s < t$, we have the general formula $Cov[W(s), W(t)] = \min\{s, t\}$

Q_2 let $0 \le s < t$ be given. What is $E[W(t) \mid W(s) = x]$

A_2: $E[W(t) \mid W(s) = x]$

$$
\begin{aligned}
&= E[W(t) - W(s) + W(s) \mid W(s) = x] \\
&= E[W(t) - W(s) \mid W(s) = x] + E[W(s) \mid W(s) = x] \\
&= E[W(t) - W(s)] + x \\
&= 0 + x \quad = x
\end{aligned}
$$

martingale property of Brownian motion

Q_3 Let $0 \le s < t$ be given, what is $E[W(s) \mid W(t) = x]$

A_3: $W(s) = \lambda W(t) + (W(s) - \lambda W(t))$

choose λ so that $\lambda W(t)$ and $W(s) - \lambda W(t)$ are independent

Then $E[W(s) \mid W(t) = x] = E[\lambda W(t) + W(s) - \lambda W(t) \mid W(t) = x]$
$= \lambda x$

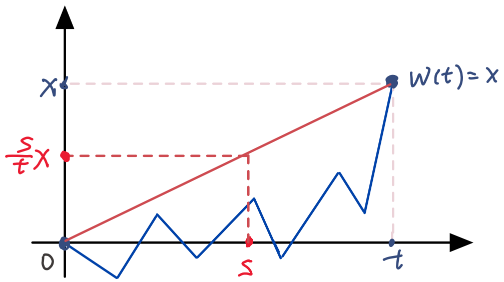
to find λ, we have

$Cov[W(t), W(s) - \lambda W(t)]$
$= E[W(t)(W(s) - \lambda W(t))] = E[W(t)W(s)] - \lambda E[W^2(t)]$
$= t \cdot s - \lambda \cdot t^2 = 0$

thus $\lambda = \frac{s}{t}$

$\therefore E[W(s) \mid W(t) = x] = \frac{sx}{t}$

to remember the result, use follow plot



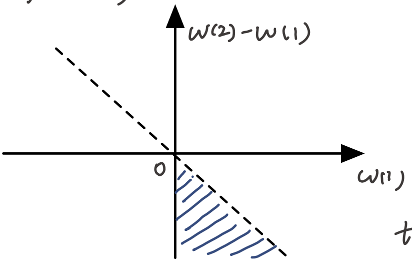
Q4: what is $P\{W(1)>0 \text{ and } W(2)<0\}$?

A4: $W(1)>0$ and $W(2)<0$

$\Leftrightarrow$ $W(1)>0$ and $W(2)-W(1)+W(1)<0$

$W(2)-W(1)$ and $W(1)$ are independent

$\Rightarrow$



joint distribution of $W(2)-W(1)$ and $W(1)$ is bivariate normal

thus $P\{W(1)>0 \text{ and } W(2)<0\}$

$$= \frac{1}{8}$$

Q5 Is $X(t) = \cosh(\lambda W(t)) e^{-\frac{1}{2}\lambda^2 t}$ a martingale?

$\left(\cosh(x) = \frac{1}{2}(e^{x} + e^{-x}) \right)$

A5: $X(t) = \frac{1}{2}\left(e^{\lambda W(t)} + e^{-\lambda W(t)}\right) e^{-\frac{1}{2}\lambda^2 t}$

$$= \frac{1}{2}\Big(\underbrace{e^{\lambda W(t) - \frac{1}{2}\lambda^2 t}}_{\text{martingale}} + \underbrace{e^{-\lambda W(t) - \frac{1}{2}\lambda^2 t}}_{\text{martingale}} \Big)$$

= Martingale

Q_6: Is $2^{W(t)}$ a martingale?

A_6: $M(t) = e^{(\log 2) W(t) - \frac{1}{2}(\log 2)^2 t}$

is a martingle

but $e^{(\log 2) W(t)} = 2^{W(t)}$ is not

Q_7: Let $S(t) = S(0) \exp[\sigma W(t) + (\alpha - \frac{1}{2}\sigma^2)t]$

what is $\lim_{t\to\infty} W(t)$, $\lim_{t\to\infty} \frac{W(t)}{t}$, $\lim_{t\to\infty} S(t)$, $\lim_{t\to\infty} E[S(t)]$

A_7: ▷ for $W(t)$, as $t \to \infty$, the paths of $W(t)$ oscillate ever more widely, thus $\lim_{t\to\infty} W(t)$ does not exist

▷ for $\lim_{t\to\infty} \frac{W(t)}{t}$ as $t \to \infty$

$$E[\tfrac{1}{t} W(t)] = 0 \qquad Var[\tfrac{1}{t} W(t)] = \tfrac{1}{t^2} Var(W(t)) = \tfrac{1}{t}$$

the randomness in $\frac{1}{t} W(t)$ disappears because its variance converges to zero

thus $\lim_{t\to\infty} \frac{W(t)}{t} = 0$

▷ for $S(t)$

if $\alpha - \frac{1}{2}\sigma^2 > 0$

$$S(t) = S(0) \exp\Big[\overbrace{t\big(\tfrac{\sigma}{t} W(t) - (\alpha - \tfrac{1}{2}\sigma^2)\big)}^{\text{positive limit}}\Big] \to \infty \quad \text{as } t \to \infty$$

if $\alpha - \frac{1}{2}\sigma^2 < 0$

$$S(t) = S(0)\exp\Big[\underbrace{t\Big[\frac{\sigma}{t}W(t) - (\alpha - \frac{1}{2}\sigma^2)\Big]}_{\text{negative limit}}\Big] \to 0 \quad \text{as } t \to \infty$$

if $\alpha - \frac{1}{2}\sigma^2 = 0$

$S(t) = S(0)e^{\sigma W(t)}$ does not have a limit

because $W(t)$ does not have a limit when $t \to \infty$

▷ for $E[S(t)]$

$= S(0)e^{\alpha t}$

if $\alpha > 0$ $\quad E[S(t)] \to \infty$ as $t \to \infty$

if $\alpha < 0$ $\quad E[S(t)] \to 0$ as $t \to \infty$

if $\alpha = 0$ $\quad E[S(t)] = S(0)$ $\forall t \geq 0$

Q8: show that when $0 \leq s < t$

$$E\Big[\big(W(s) - \frac{s}{t}x\big)^2 \,\Big|\, W(t) = x\Big] = \frac{s(t-s)}{t}$$

A8: $W(s) - \frac{s}{t}W(t)$ is independent of $W(t)$

so does $\big(W(s) - \frac{s}{t}W(t)\big)^2$

$$\therefore E\Big[\big(W(s) - \frac{s}{t}x\big)^2 \,\Big|\, W(t) = x\Big] = E\Big[\big(W(s) - \frac{s}{t}W(t)\big)^2\Big]$$

$$= E\Big[W^2(s) - \frac{2s}{t}W(s)W(t) + \frac{s^2}{t^2}W^2(t)\Big]$$

$$= s + \frac{s^2}{t} - \frac{2s}{t} E[W(s)W(t)]$$

$$E[W(s)W(t)] = E\left[\left(W(t) - W(s) + W(s)\right) W(s)\right]$$

$$= E\left[W(s)\left(W(t) - W(s)\right) + W^2(s)\right]$$

$$= s$$

$$\therefore E\left[\left(W(s) - \frac{s}{t}x\right)^2 \middle| W(t) = x\right] = s + \frac{s^2}{t} - \frac{2s^2}{t}$$

$$= s - \frac{s^2}{t}$$

$$= \frac{s(t-s)}{t}$$