Q1 Let $T>0$, N is a positive integer, define

$$t_k = \frac{kT}{N} \qquad k = 0, 1, \cdots N$$

show that

$$\sum_{j=1}^{N} \left(W(t_j) - W(t_{j-1}) \right)^2$$

has expected value T and variance $\frac{2T^2}{N}$

A1

$$E\left[\sum_{j=1}^{N} \left(W(t_j) - W(t_{j-1}) \right)^2 \right]$$

$$= \sum_{j=1}^{N} E\left[\left(W(t_j) - W(t_{j-1}) \right)^2 \right]$$

$$= \sum_{j=1}^{N} Var\left[W(t_j) - W(t_{j-1}) \right]$$

$$= \sum_{j=1}^{N} (t_j - t_{j-1}) = T$$

the normal distribution $N(0, \sigma^2)$ has the MGF

$$\varphi(u) = E[e^{ux}] = e^{\frac{1}{2}\sigma^2 u^2}$$

$$\varphi'(u) = \sigma^2 u e^{\frac{1}{2}\sigma^2 u^2} \quad \Rightarrow \varphi'(0) = 0$$

$$\varphi''(u) = \sigma^2 e^{\frac{1}{2}\sigma^2 u^2} + \sigma^4 u^2 e^{\frac{1}{2}\sigma^2 u^2} \quad \Rightarrow \varphi''(0) = \sigma^2$$

$$\varphi^{(3)}(u) = \sigma^4 u e^{\frac{1}{2}\sigma^2 u^2} + 2\sigma^4 u e^{\frac{1}{2}\sigma^2 u^2} + \sigma^6 u^3 e^{\frac{1}{2}\sigma^2 u^2}$$

$$= 3\sigma^4 u e^{\frac{1}{2}\sigma^2 u^2} + \sigma^6 u^3 e^{\frac{1}{2}\sigma^2 u^2} \quad \Rightarrow \varphi^{3}(0) = 0$$

$$\varphi^{(4)}(u) = 3\sigma^4 e^{\frac{1}{2}\sigma^2 u^2} + 3\sigma^6 u^2 e^{\frac{1}{2}\sigma^2 u^2} + 3\sigma^6 u^2 e^{\frac{1}{2}\sigma^2 u^2} + \sigma^8 u^4 e^{\frac{1}{2}\sigma^2 u^2}$$

$$\Rightarrow \varphi^{(4)}(0) = 3\sigma^4$$

thus

$$Var\left[\sum_{j=1}^{N} \left(W(t_j) - W(t_{j-1}) \right)^2 \right]$$

$$= \sum_{j=1}^{N} Var\left[\left(W(t_j) - W(t_{j-1}) \right)^2 \right]$$

$$= \sum_{j=1}^{N} \left[E\left[\left(W(t_j) - W(t_{j-1}) \right)^4 \right] - E^2\left[\left(W(t_j) - W(t_{j-1}) \right)^2 \right] \right]$$

$$= \sum_{j=1}^{N} \left[3(t_j - t_{j-1})^2 - (t_j - t_{j-1})^2 \right]$$

$$= \sum_{j=1}^{N} \left[2 \cdot (t_j - t_{j-1})^2 \right]$$

$$= 2 \sum_{j=1}^{N} \left(\frac{T}{N}\right)^2 = \frac{2T_j^2}{N}$$

Q2 Apply Itô's formula to $f(x) = \frac{1}{2}x^2$ to compute $\int_0^t W(u)\,dW(u)$

A2: $$f(W(t)) = f(W(0)) + \int_0^t f'(W(u))\,dW(u) + \frac{1}{2}\int_0^t f''(W(u))\,du$$

$$= 0 + \int_0^t W(u)\,dW(u) + \frac{1}{2}\int_0^t du$$

$$\Rightarrow \int_0^t W(u)\,dW(u) = f(W(t)) - \frac{1}{2}\int_0^t du$$

$$= \frac{1}{2} W^2(t) - \frac{1}{2} t$$

Q3: Use properties of Brownian motion to verify that $\frac{1}{2}W^2(t) - \frac{1}{2}t$ is a martingale

A3: from A2 we know $\frac{1}{2}W^2(t) - \frac{1}{2}t = \int_0^t W(u)\,dW(u)$ which is Itô integral thus $\frac{1}{2}W^2(t) - \frac{1}{2}t$ is a martingale

OR: Let $M(t) = \frac{1}{2}W^2(t) - \frac{1}{2}t$

we need to show that $E[M(t) / M(s) = x] = x$ $0 \le s < t$

$M(s) = x \Rightarrow \frac{1}{2}W^2(s) - \frac{1}{2}s = x$

thus $W(s) = \pm\sqrt{2x+s}$

thus $M(t) = \frac{1}{2}W^2(t) - \frac{1}{2}t$

$= \frac{1}{2}[W(t)-W(s)+W(s)]^2 - \frac{1}{2}t$

$= \frac{1}{2}[(W(t)-W(s))^2 + W^2(s) + 2W(s)(W(t)-W(s))]$

$- \frac{1}{2}t$

$\Rightarrow E[M(t) \mid M(s)=x]$

$= \frac{1}{2}E[(W(t)-W(s))^2 \mid M(s)=x] + \frac{1}{2}E[W^2(s) \mid M(s)=x]$

$+ E[W(s)(W(t)-W(s)) \mid M(s)=x] - \frac{1}{2}E[t]$

$= \frac{1}{2}(t-s) + \frac{1}{2}(2x+s) - \frac{1}{2}t = x$

thus $M(t) = \frac{1}{2}W^2(t) - \frac{1}{2}t$ is a martingale

Q4: Is $W^3(t)$ a martingale?

A4: let $f(x) = x^3$, the $f'(x) = 3x^2$ $f''(x) = 6x$

$W^3(t) = \int_0^t f'(W(u))\,dW(u) + \frac{1}{2}\int_0^t f''(W(u))\,du$

$= \underbrace{\int_0^t 3W^2(u)\,dW(u)}_{\text{Martingale}} + \underbrace{3\int_0^t W(u)\,du}_{\text{not a martingale}}$

$\neq$ martingale

lemma:

for an Itô integral, i.e. $\int_0^t \Delta(u)\, dW(u)$

if ① $\Delta(u)$ only depends on the path of W between 0 and u for every u

(e.g. $W(u)$ ✓ $W^2(u)$ ✓ $W(u+1)$ ✗)

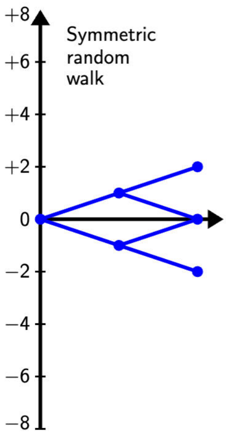
② $E\left[\int_0^T \Delta^2(u)\, du\right] < \infty$

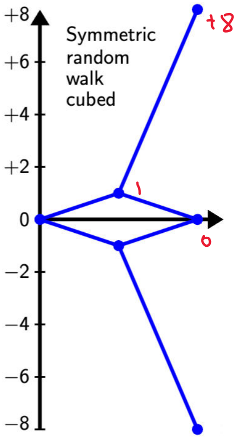
$\Rightarrow \int_0^t \Delta(u)\, dW(u)$ is a martingale

for $W^2(u)$ $E\left[\int_0^T W^4(u)\, du\right] = \int_0^T E[W^4(u)]\, du = \int_0^T 3u^2\, du < \infty$

thus $\int_0^t W^2(u)\, dW(u)$ is a martingale

OR. intuitively. martingale should satisfy $E[M(t+1) \mid M(t)] = M(t)$ for every node





$E[W^3(2)] = 4 \neq 1$

Q5: let $0 \le s < t$ be given, use distributional properties of Brownian motion to compute and show that

$$E[W^3(t) \mid W^3(s) = x] \neq x \qquad 0 \le s < t$$

A5: Since $W^3(t)$ is not a martingale

thus $E[W^3(t) \mid W^3(s) = x] \neq x$

OR: let $W(t) = W(t) - W(s) + W(s)$

calculate $W^3(t) = \left([W(t) - W(s)] + W(s)\right)^3$

$$\Rightarrow E[W^3(t) \mid W^3(s) = x]$$

$$= E[((W(t) - W(s)) + W(s))^3 \mid W^3(s) = x]$$

$$= E[(W(t) - W(s))^3 \mid W^3(s) = x] + 3E[(W(t) - W(s))^2 W(s) \mid W^3(s) = x]$$

$$+ 3E[(W(t) - W(s))W^2(s) \mid W^3(s) = x] + E[W^3(s) \mid W^3(s) = x]$$

$$= E[(W(t) - W(s))^3] + 3x^{\frac{1}{3}} E[(W(t) - W(s))^2]$$

$$+ 3x^{\frac{2}{3}} E[W(t) - W(s)] + x$$

$$= x + 3x^{\frac{1}{3}}(t-s) \neq x$$