§ Chapter 2. Brownian Motion

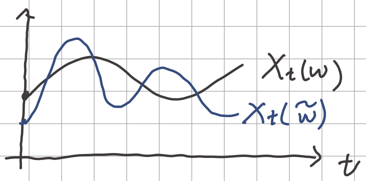
Def 1.2.4: A stochastic process $X = (X_t)_{t\geq 0}$ is a collection of random variables $\{X_t : 0 \leq t < \infty\}$

We think about X as a function

$$X : \Omega \times [0,\infty) \to \mathbb{R}$$
$$X : \omega \times t \quad \to X_t(\omega)$$

In particular,

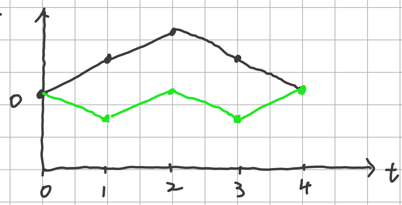
- for fixed t: $\omega \mapsto X_t(\omega)$ is a r.v
  i.e. $\mathcal{F}$-measurable $\{\omega \leq \Omega : X_t(\omega) \leq c\} \in \mathcal{F}$ $\forall c \in \mathbb{R}$
- for fixed ω $t \mapsto X_t(\omega)$ is called the sample path / trajetory of X



A process X is called continuous if its sample paths are continuous, i.e. $\lim_{s\to t} X_s(\omega) = X_t(\omega)$

Def. 2.1. (Symmetric Random Walks) Take r.v. $\xi_1, \xi_2, \ldots$ iid $P(\xi_i = 1) = P(\xi_i = -1) = 1/2$. Define $\xi_0 = 0$, $S_k = \sum_{n=1}^{k} \xi_n$ $\forall k \in \mathbb{N}$

e.g.



S_t (within adjacent integers) = linear interpolation of S_k

3. observations:

- S' has independent increments

for all $0 = k_0 < k_1 < k_2 < \cdots < k_m$ $m \in \mathbb{N}$ integers

$(S_{k_1} - S_{k_0}), (S_{k_2} - S_{k_1}), \cdots (S_{k_m} - S_{k_{m-1}})$

are independent

observe that $S_{k_{i+1}} - S_{k_i} = \sum_{n=1}^{k_{i+1}} \xi_n - \sum_{n=1}^{k_i} \xi_n$

$$= \underline{\sum_{n=k_i+1}^{k_{i+1}} \xi_n}$$ non-overlapping r.v.

- $\mathbb{E}[S_{k_{i+1}} - S_{k_i}] = \mathbb{E}\left[\sum_{n=k_i+1}^{k_{i+1}} \xi_n\right] = \sum_{n=k_i+1}^{k_{i+1}} \mathbb{E}[\xi_n] = 0$
- $\mathrm{Var}[S_{k_{i+1}} - S_{k_i}] = \mathrm{Var}\left[\sum_{n=k_i+1}^{k_{i+1}} \xi_n\right] = \sum_{n=k_i+1}^{k_{i+1}} \mathrm{Var}[\xi_n] = k_{i+1} - k_i$

$$\left(\mathrm{Var}(\xi_n) = \mathbb{E}[\xi_n^2] - \mathbb{E}[\xi_n]^2 = \tfrac{1}{2}[1^2 + (-1)^2] - 0^2 = 1 \right)$$

$\Rightarrow \mathrm{Var}(S_k) = \mathrm{Var}(S_k - S_0) = k - 0 = k$

$\mathbb{E}[S_k] = \mathbb{E}[S_k - S_0] = 0$

rescale S so that it takes a random step at shorter and shorter time intervals.

Def. 2.2. (scaled symmetric random walks)

For $\varepsilon > 0$ $S_t^{\varepsilon} := \sqrt{\varepsilon}\, S_{\frac{t}{\varepsilon}}$ random walk at time $\frac{t}{\varepsilon}$

if $\frac{t}{\varepsilon} \in \mathbb{N}$ the def 2.1 tells us that

$\mathbb{E}[S_t^{\varepsilon}] = \mathbb{E}[\sqrt{\varepsilon}\, S_{t/\varepsilon}] = \sqrt{\varepsilon}\, \mathbb{E}[S_{t/\varepsilon}] = \sqrt{\varepsilon} \cdot 0 = 0$

$$Var[S_t^\varepsilon] = Var(\sqrt{\varepsilon} \cdot S_{t/\varepsilon}) = \varepsilon \cdot Var(S_{t/\varepsilon}) = \varepsilon \cdot t/\varepsilon = t$$

From CLT, one can show that (proof omitted)

Theorem 2.1: For $t \geq 0$, the distribution of S_t^ε converges to a normal distribution with mean 0 and variance t for $\varepsilon \to 0$.

Def 2.2.1:
(Brownian Motion).

A Brownian motion w (BM) is a continuous stochastic process such that

① w has independent increments

② for $s<t$, $W_t - W_s \sim N(0, \sigma^2(t-s))$

show these properties to verify a process as BM

(Standard BM)

A standard BM is a BM for which $W_0 = 0$ and $\sigma^2 = 1$

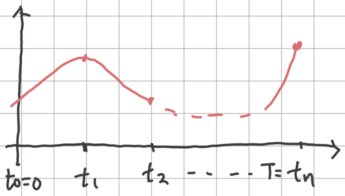
Quick Recall:

- stochastic process $\{X_t : 0 \leq t < \infty\}$
- trajectory / Sample $t \mapsto X_t(\omega)$ for fixed $\omega \in \Omega$
- X is called continuous if each trajectory is continuous
- X has stationary increments, if $\forall n \geq 0$, the distribution of $X_{t+n} - X_t$ does not depend on t (but n)

2.3. First-order variation

For now: functions $f: [0, +\infty) \to \mathbb{R}$

We take any partition $\Pi = \{t_0, t_1, \cdots, t_n\}$ of $[0, T]$: $0 = t_0 < t_1 < \cdots < t_n = T$, $n \in \mathbb{N}$



$$FV_T(f) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} |f(t_{j+1}) - f(t_j)|$$

Now take $\Pi = \{0, \frac{1}{n}, \frac{2}{n}, \cdots T\}$ $n \in \mathbb{N}$

if f is continuously differentiable, there exist $t_k^* \in [t_k, t_{k+1}]$

with $f(t_{k+1}) - f(t_k) = f'(t_k^*)(t_{k+1} - t_k)$, thus

$$\sum_{j=0}^{n-1} |f(t_{j+1}) - f(t_j)| = \sum_{j=0}^{n-1} |f'(t_j^*)| (t_{j+1} - t_j) \text{, and}$$

$$FV_T(f) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} |f'(t_j^*)| (t_{j+1} - t_j) = \int_0^T |f'(t)| dt$$

Prop 2.3.1: If W is a standard BM, then

$$\lim_{n \to \infty} \mathbb{E}\left[\sum_{k=0}^{n-1} |W_{\frac{k+1}{n}} - W_{\frac{k}{n}}| \right] = \infty$$

$\Rightarrow$ BM has first-order variation ∞ almost surely

2.4 The quadratic variation of BM

Def 2.4.1 $f: [0, +\infty) \to \mathbb{R}$, the quadratic variation of f up to time T is

$$[f, f]_T := \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} |f(t_{j+1}) - f(t_j)|^2 \quad \Pi = \{t_0, \cdots t_n\}$$

$$0 = t_0 < \cdots < t_n = T$$

Theorem. If f is continuously differentiable, then $[f,f]_T = 0$

Proof: According to the Mean-Value theorem.

$$\sum_{k=0}^{n-1}[f(t_{k+1})-f(t_k)]^2 = \sum_{k=0}^{n-1}[f'(t_k^*)(t_{k+1}-t_k)]^2$$

$$\leq \max_{0\leq t\leq T}\{[f'(t)]^2\}\cdot\sum_{k=0}^{n-1}(t_{k+1}-t_k)^2$$

$$= \max_{0\leq t\leq T}\{[f'(t)]^2\}\cdot\sum_{k=0}^{n-1}\frac{T^2}{n^2}$$

$$= \max_{0\leq t\leq T}\{[f'(t)]^2\}\frac{T^2}{n} \xrightarrow{n\to\infty} 0$$

Remark:

$$\sum_{k=0}^{n-1}(t_{k+1}-t_k)^2 \to 0 \iff dt\cdot dt = 0$$

Theorem 2.4.2. let W be standard BM. Then $[W,W]_t = t \quad \forall t \geq 0$ a.s

a random variable

$$\underbrace{\lim_{\|\Pi\|\to\infty}\sum_{j=0}^{n-1}|W_{t_{j+1}}-W_{t_j}|^2}_{\text{r.v.}}$$

Remark:

- the paths of Brownian motion cannot be differentiated w.r.t. time

- If we use BM to model the price of a stock, we do not want the path of BM to be differentiable. The existence of a derivative would permit one to see whether the price is rising or falling.

D. We also have $Var[W(T)] = T$. but it's computed by averaging over all possible paths. The quadratic variation is computed along a single path, and we get the same answer regardless of the path.

Proof.

we have $t_{k+1} - t_k = \frac{T}{n}$

$$\sum_{k=0}^{n-1}[W(t_{k+1}) - W(t_k)]^2 = T \cdot \frac{1}{n}\sum_{k=0}^{n-1}\left[\frac{W(t_{k+1}) - W(t_k)}{\sqrt{T/n}}\right]^2$$

$$= T \cdot \frac{1}{n}\sum_{k=0}^{n-1}\left[\frac{W(t_{k+1}) - W(t_k)}{\sqrt{t_{k+1} - t_k}}\right]^2$$

For each k, the random variable

$$\frac{W(t_{k+1}) - W(t_k)}{\sqrt{t_{k+1} - t_k}} \sim N(0,1)$$

thus. the r.v. $\left[\frac{W(t_{k+1}) - W(t_k)}{\sqrt{t_{k+1} - t_k}}\right]^2$ are i.i.d. having the same mean value of 1 ($E[X^2] = Var(X) + E[X]^2$)

by Law of Large Numbers.

$$\frac{1}{n}\sum_{k=0}^{n-1}\left[\frac{W(t_{k+1}) - W(t_k)}{\sqrt{t_{k+1} - t_k}}\right]^2 \to E\left(\left[\frac{W(t_{k+1}) - W(t_k)}{\sqrt{t_{k+1} - t_k}}\right]^2\right) = 1$$

thus

$$\sum_{k=0}^{n-1}[W(t_{k+1}) - W(t_k)]^2 = T \cdot \frac{1}{n}\sum_{k=0}^{n-1}\left[\frac{W(t_{k+1}) - W(t_k)}{\sqrt{t_{k+1} - t_k}}\right]^2 \to T$$

the equation is equivalent to $dW(t)\,dW(t) = dt$

Additional Theorems

<1>

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} |W(t_{k+1}) - W(t_k)|\,(t_{k+1} - t_k) = 0$$

we record this fact by writing $dW(t)\,dt = 0$

<2>

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} |W(t_{k+1}) - W(t_k)|^3 = 0$$

we record this fact by writing $(dW(t))^3 = 0$