§ Modeling Financial Markets in a B-S framework.

6.1 GBM & Hedging (2 assets)

- The Bank account B solves: $dB_t = rB_t dt$ or $B_t = B_0 e^{rt}$
- Stock price S_t:

α: mean return rate (percentage drift)
σ: percentage volatility

$$dS_t = \underbrace{\alpha S_t dt}_{\text{drift}} + \underbrace{\sigma S_t dW_t}_{\text{diffusion (noisy fluctuations)}}$$

Def 6.1: S_t is called a geometric BM (gBM) if

$$dS_t = \alpha S_t dt + \sigma S_t dW_t, \text{ and thus}$$

$$S_t = S_0 e^{(\alpha - \frac{\sigma^2}{2})t + \sigma W_t}$$

Now, invest in (B, S), denote the portfolio value by $(X_t)_{t \geq 0}$ the initial capital is X_0. At each time step t, we have Δ_t shares stocks and Γ_t balance in bank account.

$(*)$ $X_t = \Delta_t S_t + \Gamma_t B_t$

$(**)$ $dX_t = \Delta_t dS_t + \Gamma_t dB_t \Rightarrow$ self-balancing (not putting or taking money out of portfolio)

Solving $(*)$ we have $\Gamma_t = (X_t - \Delta_t S_t)/B_t$

Plug it into $(**)$

$$dX_t = \Delta_t dS_t + \frac{(X_t - \Delta_t S_t)}{B_t} dB_t$$

$$= \Delta_t (\alpha S_t dt + \sigma S_t dW_t) + r(X_t - \Delta_t S_t) dt$$

$$= (\underbrace{rX_t}_{\text{average return}} + \underbrace{(\alpha - r)\Delta_t S_t}_{\text{risk premia for investing in stocks}}) dt + \underbrace{\sigma \Delta_t S_t}_{\text{Volatility term}} dW_t$$

Pricing Options

Def 6.1.2 The arbitrage-free price of an option with payoff V_T at maturity T is the value of a portfolio X satisfying $X_T = V_T$. The portfolio is called the hedging portfolio or replicating portfolio.

Example: Call option with maturity T, strike K

$$V_T = (S_T - K)^+$$

6.2. The Black-Scholes PDE

partial differential equation

The arbitrage-free price of a call option with payoff $V_T = (S_T - K)^+$ only depends on S_t, $T-t$, σ, r, not α (drift term)

Theorem 6.2.1 Consider a market with asset (B, S) $V_T = (S_T - K)^+$

① Assume that the arbitrage-free price of the call option is $c(t, S_t)$ for some function $(t, x) \to c(t, x)$. Then c satisfies the Black-Scholes PDE

$$\partial_t c + r \cdot x \, \partial_x c + \frac{\sigma^2 x^2}{2} \partial_x^2 c - rc = 0 \qquad x > 0 \quad t < T$$

boundary conditions
$$c(t, 0) = 0 \qquad t \leq T$$
$$c(t, x) = (x - K)^+ \qquad x \geq 0$$

② Conversely, if c satisfies the BS PDE, then $c(t, S_t)$ is the arbitrage-free price of the call option.

The above PDE can be solved

$$c(t,x) = x\Phi(d_+(T-t,x)) - ke^{-r(T-t)}\Phi(d_-(T-t,x))$$

where $x > 0$, $0 \le t < T$ and

$$d_\pm(u,x) := \frac{1}{\sigma\sqrt{u}}\left[\log\left(\frac{x}{k}\right) + \left(r \pm \frac{\sigma^2}{2}\right)u\right]$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}\,dy = \int_{-\infty}^{x}\varphi(y)\,dy$$

$$\varphi(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}$$

proof of theorem 6.2.1 follows from using dX_t and Itô's formula for $c(t,S_t)$. In particular, this gives you the Delta-hedging rule $\Delta_t = \partial_x c(t,S_t)$

Put-call parity: Put option $V_T = (K-S_T)^+$

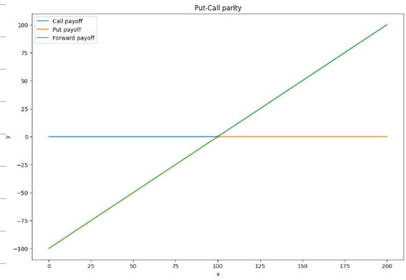
Put price $p(t,S_t)$

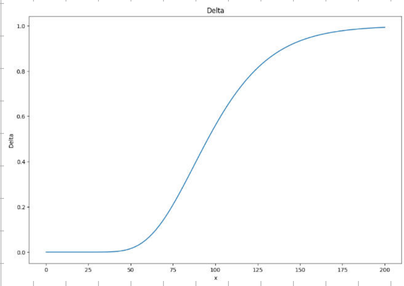
Note that $X_T = \underbrace{(S_T-K)^+}_{\text{one long call}} - \underbrace{(K-S_T)^+}_{\text{one short put}} = \overbrace{S_T}^{\text{1 share stock}} - \underbrace{K}_{\text{cash}}$

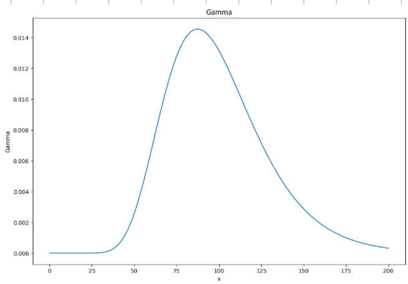
$$\Rightarrow \quad X_t = c(t,S_t) - p(t,S_t) = S_t - Ke^{-r(T-t)}$$

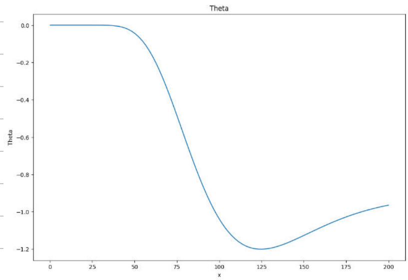
Greeks: partial derivatives of c w.r.t. t and x measure sensitivity of c w.r.t. change in either t or x, holding all other things unchanged

- Delta: $\partial_x c = \Phi(d_+) \geq 0$
- Gamma: $\partial_x^2 c = \frac{1}{x\sigma\sqrt{2\pi(T-t)}}\exp\left(-\frac{1}{2}d_+^2\right) \geq 0$
- Theta: $\partial_t c = -rKe^{-r(T-t)}N(d_-) - \frac{\sigma x}{2\sqrt{T-t}}\varphi(d_+) < 0$









Prop 6.2.1: The function $(t,x) \to c(t,x)$ is convex increasing as a function of x and it is decreasing as a function of t.