§ Risk Neutral Pricing

Pricing Securities $\rightarrow$ PDE & Hedging $\nearrow$ Itô's formul

Pricing Securities $\rightarrow$ Girsanov Theorem (change of measure)

Change of measure & Girsanov's Theorem.

Def 7.1.1 (The Girsanov Theorem)

Two probability measures $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are said to be equivalent if for $\forall A \in \mathcal{F}$, we have $\mathbb{P}(A)=0 \Leftrightarrow \widetilde{\mathbb{P}}(A)=0$

for example,

Let Z be a random variable, $Z>0$ and $E[Z]=1$. Define

$$\widetilde{\mathbb{P}}(A) = E[Z \mathbb{1}_A] = \int_A Z \, d\mathbb{P} \quad \forall A \in \mathcal{F} \quad (*)$$

Then $\widetilde{\mathbb{P}}$ is a probability measure, equivalent to $\mathbb{P}$.

Remark:

The assumption $E[Z]=1$ is required to guarantee $\widetilde{\mathbb{P}}(\Omega)=1$

Def 7.1.2. If $\widetilde{\mathbb{P}}$ is defined as $(*)$, then we write

$$d\widetilde{\mathbb{P}} = Z \, d\mathbb{P} \quad , \quad Z = \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} \quad \text{for } (*)$$

and Z is called the density of $\widetilde{\mathbb{P}}$ w.r.t. $\mathbb{P}$.

Example: $\Omega = \{\omega_1, \omega_2\}$ $\mathbb{P}(\{\omega_1\}) = \frac{1}{2} = \mathbb{P}(\{\omega_2\})$

$$Z(\omega_1) = \frac{2}{3} \quad Z(\omega_2) = 4/3 \quad \Rightarrow \widetilde{\mathbb{P}} = ?$$

$$\widetilde{\mathbb{P}}(\{\omega_1\}) = \mathbb{E}[\mathbb{1}_{\{\omega_1\}} Z]$$

$$= Z(\omega_1)\cdot \underbrace{1_{\{\omega_1\}}}_{=1} P(\{\omega_1\}) + Z(\omega_2)\cdot \underbrace{1_{\{\omega_2\}}}_{=0} P(\{\omega_2\})$$

$$= Z(\omega_1)\cdot P(\{\omega_1\}) = \frac{2}{3}\times\frac{1}{2} = \frac{1}{3}$$

$$\tilde{P}(\{\omega_2\}) = E[1_{\{\omega_2\}} Z]$$

$$= Z(\omega_2)\cdot P(\{\omega_2\}) = \frac{4}{3}\times\frac{1}{2} = \frac{2}{3}$$

Proof of lemma ($\tilde{P}$ is a prob measure, and is equivalent to P)

① $\tilde{P}$ is a probability measure $\tilde{P}(\Omega)=1$

$\tilde{P}(\Omega) = E[Z\cdot 1_\Omega] = E[Z] = 1$

Let $A_1, \dots A_N \in \mathcal{F}$ disjoint. Then

$$\tilde{P}(A_1 \cup \dots \cup A_N) = \tilde{P}(A_1) + \dots + \tilde{P}(A_n)$$

$$\| $$

$$E[Z \underbrace{1_{\{A_1 \cup \dots \cup A_N\}}}_{=1_{A_1}+1_{A_2}+\dots+1_{A_N} \text{ (disjoint)}}] = E[Z1_{A_1} + \dots + Z1_{A_N}]$$

$$= E[Z1_{A_1}] + \dots + E[Z1_{A_N}]$$

$$= \tilde{P}(A_1) + \dots + \tilde{P}(A_N)$$

$\tilde{P}(A) \in [0,1]$

$\|$

$E[\underset{\geq 0}{Z}\, \underset{\geq 0}{1_A}] \geq 0 \qquad E[\underset{\geq 0}{Z}\, \underset{\leq 1}{1_A}] \leq E[Z] = 1$

② Fix $A \in \mathcal{F}$

assume $P(A) = 0$ since $P(A) = E[\underset{\geq 0}{1_A}] = 0$, thus $1_A = 0$

$\Rightarrow E[Z\cdot 1_A] = 0$, that is $\tilde{P}(A) = 0$

assume $\tilde{P}(A) = 0 \Rightarrow Z\cdot 1_A = 0 \Rightarrow 1_A = 0 \Rightarrow E[1_A] = P(A) = 0$

$\|$

$E[Z\cdot \underset{\geq 0}{1_A}] = 0$

Special choice of Z

Suppose $T>0$ is fixed, and Z is a MG such that $Z_T>0$ and $E[Z_T]=1$

Define a new measure $\tilde{\mathbb{P}}$ via $d\tilde{\mathbb{P}} = Z_T dP$. We will denote expectation and conditional expectations w.r.t. $\tilde{\mathbb{P}}$ by $\tilde{\mathbb{E}}$.

i.e. $\tilde{\mathbb{E}}[X]$, $\tilde{\mathbb{E}}[X|\mathcal{G}]$

In particular, $\forall$ r.v. X. $\tilde{\mathbb{E}}[X] = \mathbb{E}[Z_T X] = \int Z_T X \, d\mathbb{P}$

Theorem 7.1.3. (Cameron, Martin, Girsonov)

Let $(b_t)_{t\geq 0}$ be an adopted process, W, be a SBM. and define

$$\tilde{W}_t := W_t + \int_0^t b_s \, ds$$

Let Z be the process

for this chapter, Z is of this form, and is a M.G.

$$Z_t := \exp\left(-\int_0^t b_s \, dW_s - \frac{1}{2}\int_0^t b_s^2 \, ds\right)$$

and define a new measure $d\tilde{\mathbb{P}} = Z_T d\mathbb{P}$.

(under certain conditions)

In our setting, Z is a MG, and $\tilde{W}$ is a BM under $\tilde{\mathbb{P}}$.

Note that $Z_0 = 1$ and $\mathbb{E}[Z_T]=1$

Proof: denote $M_t := \int_0^t b_s \, dW_s$ i.e. $dM_t = b_t \, dW_t$

$$f(t,x) = \exp\left(-x - \frac{1}{2}\int_0^t b_s^2 \, ds\right)$$

So, $Z_t = f(t, M_t)$

$$\Rightarrow \quad \partial_t f(t,x) = f(t,x)\left(-\frac{1}{2}b_t^2\right)$$

$$\partial_x f(t,x) = f(t,x)(-1)$$

$$\partial_x^2 f(t,x) = f(t,x), \quad [M,M]_t = \int_0^t b_s^2 \, ds$$

Itô's formula leads to

$$dZ_t = df(t, M_t) = \partial_t f\, dt + \partial_x f\, dM_t + \frac{1}{2}\partial_x^2 f\, d[M,M]_t$$

$$= Z_t\left(-\frac{1}{2} b_t^2 dt - b_t dW_t + \frac{1}{2} b_t^2 dt\right)$$

$$= -Z_t b_t dW_t$$

$\Rightarrow Z_t$ is a MG.

Remark:

$\widetilde{\mathbb{E}}[X] = \mathbb{E}[XZ]$, however $\widetilde{\mathbb{E}}[X|\mathcal{G}] \neq \mathbb{E}[XZ|\mathcal{G}]$

Lemma 7.1.7: (Bayes Theorem)

Let X be a r.v. and $d\widetilde{\mathbb{P}} = Z_T d\mathbb{P}$. let $\mathcal{G}$ be a σ-algebra, $\mathcal{G} \subseteq \mathcal{F}$

Then,

$$\widetilde{\mathbb{E}}[X|\mathcal{G}] = \frac{\mathbb{E}[Z_T X|\mathcal{G}]}{\mathbb{E}[Z_T|\mathcal{G}]}$$

In particular, if $\mathcal{G} = \mathcal{F}_s$ (filtration) and Z is defined as above. Let $0 \leq s \leq t \leq T$. If X is a $\mathcal{F}_t$-measurable r.v. then

$$\widetilde{\mathbb{E}}[X|\mathcal{F}_s] = \frac{1}{Z_s}\mathbb{E}[Z_t X|\mathcal{F}_s]$$

Lemma 7.1.2. An adapted process M is a martingale under $\widetilde{\mathbb{P}}$ if and only if MZ is a martingale under $\mathbb{P}$.

Risk Neutral Pricing

Recall $dS_t = \alpha_t S_t dt + \sigma_t S_t dW_t$ (stock price modeled by a generalized GBM).

$dB_t = B_t R_t dt$, i.e. $B_t = B_0 \exp\left(\int_0^t R_s ds\right)$

$B_0 = 1$

$B_t D_t = B_0$ is always a M.G.

Define $D_t = \exp\left(-\int_0^t R_s ds\right)$ the discount process.

Def 7.2.1. (Risk-neutral measure). A risk-neutral measure $\widetilde{\mathbb{P}}$

is a probability measure satisfying

- $\forall A \in \mathcal{F}$, $\mathbb{P}(A) = 0 \Leftrightarrow \tilde{\mathbb{P}}(A) = 0$
- under $\tilde{\mathbb{P}}$, $D_t S_t$ is a MG.

Here, we try to find $\tilde{\mathbb{P}}$ using Girsenov's theorem.

$$d(S_t D_t) = S_t dD_t + D_t dS_t + \underbrace{d[S,D]_t}_{0}$$

$\to$ 0, because D has finite Variation

$$= -R_t S_t D_t dt + D_t d(\alpha S_t dt + \sigma S_t dW_t)$$

$$= (\alpha_t - R_t) D_t S_t dt + D_t S_t \sigma_t dW_t$$

$$= \underbrace{\frac{\alpha_t - R_t}{\sigma_t}}_{\theta_t} \cdot \sigma_t D_t S_t dt + D_t S_t \sigma_t dW_t$$

$$= \sigma_t D_t S_t (\theta_t dt + dW_t)$$

where $\theta_t = \dfrac{\alpha_t - R_t}{\sigma_t}$ is the market price of risk.

Define a new process, $d\tilde{W}_t = dW_t + \theta_t dt$, and observe

$$d(S_t D_t) = \sigma_t D_t S_t d\tilde{W}_t \quad (*)$$

and $\tilde{W}_t$ is a BM under $\tilde{\mathbb{P}}$ by Girsanov's theorem.

Prop. 7.2.1. Set

$$Z_t = \exp\left(-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right)$$

Then $d\tilde{\mathbb{P}} = Z_T d\mathbb{P}$ is a risk-neutral measure.

Proof: By Girsanov theorem, $\tilde{W}$ is a BM under $\tilde{\mathbb{P}}$, and thus by $(*)$, $D_t S_t$ is a MG. under $\tilde{\mathbb{P}}$.

Theorem 7.2.2. (Risk-neutral pricing formula)

Let V_T be a $\mathcal{F}_T$-measurable r.v. Let $\tilde{\mathbb{P}}$ be our risk-neutral measure from prop 7.2.1. Then the arbitrage free price of the option with payoff V_T at $T>0$ is given by

$$V_t = \widetilde{\mathbb{E}}\left[\exp\left(-\int_t^T R_s ds\right)\cdot V_T \mid \mathcal{F}_t\right]$$

Proof:

Note that under $\widetilde{\mathbb{P}}$, $D_t V_t$ is a M.G. (prop 7.2.1)

$$D_t V_t = \widetilde{\mathbb{E}}[D_T V_T \mid \mathcal{F}_t]$$

$$\Rightarrow \quad V_t = \frac{1}{D_t}\widetilde{\mathbb{E}}[D_T V_T \mid \mathcal{F}_t]$$

$$= \widetilde{\mathbb{E}}\left[\frac{D_T}{D_t} V_T \mid \mathcal{F}_t\right]$$

$$= \widetilde{\mathbb{E}}\left[\exp\left(-\int_0^T R_s ds + \int_0^t R_s ds\right)\cdot V_T \mid \mathcal{F}_t\right]$$

$$= \widetilde{\mathbb{E}}\left[\exp\left(-\int_t^T R_s ds\right) V_T \mid \mathcal{F}_t\right]$$

Dynamics of S_t under $\widetilde{\mathbb{P}}$:

$$dS_t = \alpha_t S_t dt + \sigma_t S_t dW_t$$

$$= (\alpha_t S_t - \underbrace{\sigma_t S_t \theta_t}_{S_t(\alpha_t - R_t)})dt + \sigma_t S_t \underbrace{d\widetilde{W}_t}_{= dW_t + \theta_t dt}$$

$$= (\alpha_t S_t - \alpha_t S_t + R_t S_t)dt + \sigma_t S_t d\widetilde{W}_t$$

$$= R_t S_t dt + \sigma_t S_t d\widetilde{W}_t \quad (**)$$

Since under $\widetilde{\mathbb{P}}$, $\widetilde{W}$ is a BM, so S is a GBM with drift R_t

$\Rightarrow$ this is the reason why BS-call price does not include α.

Lemma 7.2.1. Let Δ be an adapted process and let X_t be the wealth process of self-financing portfolio that holds Δ_t shares of stock. Then $D_t X_t$ is a M.G. under $\widetilde{\mathbb{P}}$.

proof: from the self-financing condition: (6.1.7)

$$dX_t = \Delta_t dS_t + R_t(X_t - \Delta_t S_t)dt$$

continuous version.
for the discrete version, it's naive
that $X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$

using $(**)$,

$$dX_t = \Delta_t \sigma_t S_t d\widetilde{W}_t + R_t X_t dt$$

Thus, by product rule,

$$d(D_t X_t) = X_t dD_t + D_t dX_t + d[D, X]_t$$

$$= -R_t X_t D_t dt + D_t \Delta_t \sigma_t S_t d\widetilde{W}_t$$
$$+ D_t R_t X_t dt + 0$$
$$= D_t \Delta_t \sigma_t S_t d\widetilde{W}_t$$

$\Rightarrow D_t X_t$ is a MG under $\widetilde{\mathbb{P}}$.

proof of theorem 7.2.2.

Suppose X_t is the wealth process of the replicating portfolio at time t. Then by definition $V_t = X_t$ and by lemma 7.2.1. $D_t X_t$ is a MG under $\widetilde{\mathbb{P}}$, thus

$$V_t = X_t = \frac{1}{D_t} D_t X_t = \frac{1}{D_t} \widetilde{\mathbb{E}}[D_T X_T \mid \mathcal{F}_t]$$
$$= \widetilde{\mathbb{E}}\left[\frac{D_T}{D_t} X_T \mid \mathcal{F}_t\right]$$
$$= \widetilde{\mathbb{E}}\left[\frac{D_T}{D_t} V_T \mid \mathcal{F}_t\right]$$

7.3. Black-Scholes Formula

Dynamics under $\widetilde{\mathbb{P}}$

$$dS_t = S_t(r\,dt + \sigma\, d\widetilde{W}_t) \qquad \sigma, r \text{ fixed and } > 0$$

$$dB_t = rB_t\, dt \qquad \widetilde{W}_t \text{ is a BM under } \widetilde{\mathbb{P}}$$

$$\Rightarrow S_T = S_0 e^{(r-\frac{\sigma^2}{2})T + \sigma \widetilde{W}_T}$$

$$= S_0 e^{(r-\frac{\sigma^2}{2})t + \sigma\widetilde{W}_t} \cdot e^{(r-\frac{\sigma^2}{2})(T-t) + \sigma(\widetilde{W}_T - \widetilde{W}_t)}$$

$$= \underbrace{S_t}_{\mathcal{F}_t\text{-measurable}} \cdot e^{(r-\frac{\sigma^2}{2})(T-t) + \sigma\underbrace{(\widetilde{W}_T - \widetilde{W}_t)}_{\text{independent of } \mathcal{F}_t}}$$

Arbitrage-free price of European call using the Risk-neutral pricing formula.

$$V_t = \widetilde{\mathbb{E}}[(S_T - K)^+ e^{-r(T-t)} \mid \mathcal{F}_t]$$

$$= e^{-r(T-t)}\widetilde{\mathbb{E}}[(S_T-K)^+ \mid \mathcal{F}_t]$$

$$= e^{-r(T-t)}\widetilde{\mathbb{E}}[(S_t \cdot e^{(r-\frac{\sigma^2}{2})(T-t) + \underbrace{\sigma(\widetilde{W}_T - \widetilde{W}_t)}_{\perp \mathcal{F}_t}} - K)^+ \mid \mathcal{F}_t]$$

independence lemma

$$= e^{-r(T-t)} \int_{-\infty}^{+\infty} (S_t e^{(r-\frac{\sigma^2}{2})(T-t) + \sigma\sqrt{T-t}\cdot y} - K)^+ \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}}_{\text{pdf of standard normal}} dy$$

Set $S_t = x$ and define

$$d_\pm(\tau, x) = \frac{1}{\sigma\sqrt{\tau}}\left(\ln\left(\frac{x}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right)$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy = \frac{1}{\sqrt{2\pi}}\int_{-x}^{\infty} e^{-\frac{y^2}{2}} dy$$

therefore

$$C(t,x) = \frac{1}{\sqrt{2\pi}}\int_{-d_-}^{\infty} x \cdot \exp\left(-\frac{\sigma^2}{2}\tau + \sigma\sqrt{\tau}\cdot y - \frac{y^2}{2}\right) dy - e^{-r\tau} K \Phi(d_-)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-d_-}^{\infty} x \cdot \exp\left(-\frac{(y-\sigma\sqrt{\tau})^2}{2}\right) dy - e^{-r\tau} K \Phi(d_-)$$

$$= x \Phi(d_+) - e^{-r\tau} K \Phi(d_-)$$

$\Rightarrow$ the arbitrage-free price of European call is

$$V_t = S_t \Phi(d_+) - e^{-r(T-t)} K \Phi(d_-)$$

Similarly,

The risk neutral pricing formula says that the price of a European put is

$$P(t, S_t) = \widetilde{\mathbb{E}}[e^{-r(T-t)} (K - S_T)^+ \mid \mathcal{F}_t]$$

Set $\tau = T - t$

$$\Rightarrow P(t, S_t) = \widetilde{\mathbb{E}}[e^{-r\tau} (K - S_T)^+ \mid \mathcal{F}_t].$$

In the B-S market: $S_t = S_0 \exp\left((r - \frac{\sigma^2}{2})t + \sigma \widetilde{W}_t\right)$

where $\widetilde{W}$ is a BM under $\widetilde{\mathbb{P}}$, then

$$p(t, S_t) = e^{-r\tau} \widetilde{\mathbb{E}}[(K - S_0 \exp((r - \frac{\sigma^2}{2})T + \sigma \widetilde{W}_T))^+ \mid \mathcal{F}_t]$$

$$= e^{-r\tau} \widetilde{\mathbb{E}}[(K - S_t \exp((r - \frac{\sigma^2}{2})\tau + \sigma(\widetilde{W}_T - \widetilde{W}_t)))^+ \mid \mathcal{F}_t]$$

$$= \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{\mathbb{R}} (K - S_t \cdot \exp((r - \frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau} \cdot y))^+ \cdot e^{-\frac{y^2}{2}} dy$$

Set $S_t = x$, define

$$d_\pm(\tau, x) := \frac{1}{\sigma\sqrt{\tau}} \left(\ln(\frac{x}{K}) + (r \pm \frac{\sigma^2}{2})\tau\right), \text{ and}$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy = \frac{1}{\sqrt{2\pi}} \int_{-x}^{\infty} e^{-\frac{y^2}{2}} dy$$

then

$$p(t, x) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{\mathbb{R}} (K - x \cdot \exp((r - \frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau} \cdot y))^+ e^{-\frac{y^2}{2}} dy$$

$$= \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{-\infty}^{-d_-} (K - x \cdot \exp((r - \frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau} \cdot y)) \cdot e^{-\frac{y^2}{2}} dy$$

$$= K\cdot e^{-r\tau}\cdot\Phi(-d_-) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-d_-} x\cdot\exp\left(-\frac{\sigma^2}{2}\tau + \sigma\sqrt{\tau}\cdot y) - \frac{y^2}{2}\right)dy$$

$$= K\cdot e^{-r\tau}\Phi(-d_-) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-d_-} \cdot x\cdot\exp\left(-\frac{(y-\sigma\sqrt{\tau})^2}{2}\right)dy$$

$$= K\cdot e^{-r\tau}\Phi(-d_-) - \frac{x}{\sqrt{2\pi}}\int_{-\infty}^{-d_+}\exp\left(-\frac{y'^2}{2}\right)dy'$$

$$= K\cdot e^{-r\tau}\Phi(-d_-) - x\cdot\Phi(-d_+)$$

Remark:

BS option prices depend on

- time to maturity: $\tau = T-t$
- strike price: K
- (constant) interest rate: r
- price of the underlying asset: S_t
- (constant) volatility: σ

Hedging a short call

Suppose we sell a call with value $c(t,x)$. We want to build a replicating Portfolio

$\Rightarrow$ invest $\partial_x c$ into the asset S_t and put the rest into money market account B.

$c(t,x) - \partial_x c$

$= x\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-) - x\Phi(d_+)$

$= -Ke^{-r(T-t)}\Phi(d_-) \leq 0 \rightarrow$ to hedge the call, we will have to borrow money.

For $t \to T$

$$d_+ = \frac{1}{\sigma\sqrt{T-t}}\left(\ln\left(\frac{S_t}{K}\right) + \left(r+\frac{\sigma^2}{2}\right)(T-t)\right)$$

$$= \frac{1}{\sigma}\left(\underbrace{\frac{1}{\sqrt{T-t}}\ln\left(\frac{S_t}{K}\right)}_{\begin{cases} >0 \text{ if } S_t>K \\ \leq 0 \text{ if } S_t \leq K \end{cases}} + \underbrace{\left(r+\frac{\sigma^2}{2}\right)\cdot\sqrt{T-t}}_{\to 0 \text{ as } t\to T}\right)$$

$$= \begin{cases} +\infty & \text{if } S_t > K \\ -\infty & \text{if } S_t \leq K \end{cases}$$

thus $\Phi(d_+) = \begin{cases} 1 & \text{if } S_t > K \\ 0 & \text{if } S_t \leq 0 \end{cases}$

that is, if $S_t > K$, we will invest S_t when $t \to T$. if $S_t \leq K$, we will not invest in S_t when $t \to T$.

An example of arbitrage (?)

Assume the stock price is x_0 & we short $\partial_x C(t, x_0)$ shares of stock & buy a call valued $C(t, x_0)$, we invest

$$M = x_0 \partial_x C(t, x_0) - C(t, x_0)$$

what happens if stock price changes from x_0 to x ?

The portfolio value is

$$C(t, x) - \partial_x C(t, x_0)\cdot x + M$$

$$= C(t,x) - \partial_x C(t, x_0)\cdot x + x_0 \partial_x C(t, x_0) - C(t, x_0)$$

$$= C(\underset{\sim}{t}, x) - C(\underset{\sim}{t}, x_0) - \partial_x C(\underset{\sim}{t}, x_0)(x - x_0) \quad \geq 0$$

this inequality is wrong because we pretend t to be constant